SOLUTION OF MIXED PROBLEMS ON THE TORSION OF AN INFINITE ELASTIC CYLINDER BY THE METHOD OF DUAL INTEGRAL EQUATIONS AND BY DIFFERENTIATION OF THE BOUNDARY CONDITIONS

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Two mixed problems on the torsion of an elastic infinite continuous circular cylinder loaded symmetrically with respect to its axis are solved. In the first problem the displacement on a finite portion of the surface of the cylinder and the shear stress outside this area are given. The quantities given in the second problem are the shear stress on a finite portion of the surface of the cylinder and the displacement outside this area. In each case both symmetrical and antisymmetrical deformation with respect to a plane perpendicular to the axis of the cylinder are considered.

In solving these problems we make use of a particular solution of the torsion equation for shafts of variable cross section containing one arbitrary harmonic function, and of the method of differentiating the boundary conditions. The solutions of the problems on the torsion of the cylinder are reduced to two types of dual integral equations. Their solutions are represented in the form of integrals containing an unknown function which is found from Fredholm's integral equation of the second type with a symmetrical kernel.

1. Representation of the solution of the torsion equation. It is easy to see that the torsion equation

$$\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv)\right] = 0 \tag{1.1}$$

is satisfied by the function

$$v = \frac{\partial \delta}{\partial r} \qquad \left(\frac{\partial^2 \delta}{\partial r^2} + \frac{1}{r}\frac{\partial \delta}{\partial r} + \frac{\partial^2 \delta}{\partial z^2} = 0\right) \tag{1.2}$$

where δ is an arbitrary harmonic function. The formulas of Hooke's law then give us the following expressions for the stresses $\tau_{r\varphi}$ and $\tau_{\varphi z}$:

$$\tau_{r\varphi} = G\left[\frac{\partial^2\delta}{\partial r^2} - \frac{1}{r}\frac{\partial\delta}{\partial r}\right], \qquad \tau_{\varphi z} = G\frac{\partial^2\delta}{\partial r\,\partial z} \tag{1.3}$$

We take the harmonic function δ in the form of an improper integral,

$$\delta = \int_0^{+\infty} I_0(\lambda r)\,[A_1(\lambda)\sin\lambda z + A_2(\lambda)\cos\lambda z]\,d\lambda \tag{1.4}$$

where $A_1(\lambda)$ and $A_2(\lambda)$ are unknown functions that must be determined from the boundary conditions of the problem; $I_0(\lambda r)$ is a modified Bessel function of zero order. Substituting (1.4) into (1.2) and (1.3), we obtain

$$\begin{aligned} v &= \int_0^{+\infty} \lambda I_1(\lambda r)\,[A_1(\lambda)\sin\lambda z + A_2(\lambda)\cos\lambda z]\,d\lambda \\ \tau_{r\varphi} &= G\int_0^{+\infty} \lambda^2 I_2(\lambda r)\,[A_1(\lambda)\sin\lambda z + A_2(\lambda)\cos\lambda z]\,d\lambda \\ \tau_{z\varphi} &= G\int_0^{+\infty} \lambda^2 I_1(\lambda r)\,[A_1(\lambda)\cos\lambda z - A_2(\lambda)\sin\lambda z]\,d\lambda \end{aligned} \tag{1.5}$$

where $I_1(\lambda r)$ and $I_2(\lambda r)$ are Bessel functions and G is the shear modulus. Formulas (1.5) will be used below to solve boundary value problems on the torsion of an infinite circular cylinder.

2. Method of differentiation of the boundary conditions. Let us consider the mixed problem of elasticity theory for a symmetrically loaded solid of revolution or for a solid in a state of plane deformation or plane stress. Let the elastic solid under consideration be bounded by the surface S. We assume that on some portion S_0 of the surface S we are given a boundary condition for the projection of the displacement vector. We denote this projection by v. The process of solving mixed boundary value problems of elasticity theory can be simplified substantially in some cases if the boundary condition for v is replaced by a boundary condition for the partial derivative along the tangent. The boundary condition for v is satisfied to within a constant with the aid of the boundary condition for the partial derivative. Hence, the boundary condition for the partial derivative must be combined with a condition for v at some point of the area S_0. As the supplementary condition one can stipulate that the principal vector and the principal moment stresses applied to the area S_0 are equal to certain given values.

This method will be used in the solution of the boundary value problems considered below.

3. Representation of the solution of two types of dual integral equations. The boundary value problems to be considered are reducible to the following dual integral equations:

$$\int_0^{+\infty} [1 - g(\lambda)]\,f(\lambda)\sin\lambda z\,d\lambda = F_1(z) \qquad (0 < z < a) \tag{3.1}$$

$$\int_0^{+\infty} f(\lambda) \cos \lambda z \, d\lambda = F_2(z) \qquad (a < z < +\infty)$$

where $f(\lambda)$ is an unknown function, and $g(\lambda)$, $F_1(z)$, and $F_2(z)$ are given functions. We assume that the function $g(\lambda)$ has the following property: it is continuous in the interval $0 < \lambda < +\infty$, the product $\lambda g(\lambda)$ as $\lambda \to 0$ is finite, and there exists an α such that

$$\lim_{\lambda\to\infty} \lambda^p \, [\lambda g(\lambda) - \alpha] = 0 \qquad (0 < p < 1) \tag{3.2}$$

The above condition will henceforth be referred to as 'Condition A'. The function $F_2(z)$ is assumed representable by means of a Fourier integral.

$$F_2(z) = \int_0^{+\infty} f_2(\lambda) \cos \lambda z \, d\lambda, \qquad f_2(\lambda) = \frac{2}{\pi} \int_a^{+\infty} F_2(z) \cos \lambda z \, dz \tag{3.3}$$

By introducing the new unknown function

$$f_1(\lambda) = f(\lambda) - f_2(\lambda) \tag{3.4}$$

we see that by virtue of (3.3) equations (3.1) may be reduced to

$$\begin{gathered} \int_0^{+\infty} [1 - g(\lambda)] \, f_1(\lambda) \sin \lambda z \, d\lambda = F(z) \qquad (0 < z < a) \\ \int_0^{+\infty} f_1(\lambda) \cos \lambda z \, d\lambda = 0 \qquad (a < z < +\infty) \end{gathered} \tag{3.5}$$

where

$$F(z) = F_1(z) - \int_0^{+\infty} [1 - g(\lambda)] \, f_2(\lambda) \sin \lambda z \, d\lambda \tag{3.6}$$

We seek a solution of the dual integral equations (3.5) in the form

$$f_1(\lambda) = \int_0^a t^{1/2} \varphi(t) \, J_0(\lambda t) \, dt \tag{3.7}$$

where $\varphi(t)$ is the new function sought. Then by virtue of the integrals

$$\begin{gathered} \int_0^{+\infty} J_0(\lambda t) \sin \lambda z \, d\lambda = \begin{cases} 0, & \text{if} \quad z < t \\ (z^2 - t^2)^{-1/2}, & \text{if} \quad x > t \end{cases} \\ \int_0^{+\infty} J_0(\lambda t) \cos \lambda z \, d\lambda = \begin{cases} (t^2 - z^2)^{-1/2}, & \text{if} \quad 0 \leqslant z < t \\ 0, & \text{if} \quad t < z \end{cases} \end{gathered} \tag{3.8}$$

the second equation of (3.5) is satisfied identically. Following the procedure employed in

[1] we rewrite the first equation of (3.5) as

$$\int_0^{+\infty} f_1(\lambda) \sin \lambda z \, d\lambda = F_3(z) \qquad (0 < z < a) \tag{3.9}$$

where

$$F_3(z) = F(z) + \int_0^{+\infty} g(\lambda) f_1(\lambda) \sin \lambda z \, d\lambda \tag{3.10}$$

Assuming the right-hand side of equation (3.9) to be a known function, we substitute into it function (3.7) and once again make use of integral (3.8). Equation (3.9) is thereby reduced to the Schloemilch equation [2]

$$\frac{2}{\pi} \int_0^z \frac{t^{1/2} \varphi(t) \, dt}{\sqrt{z^2 - t^2}} = \frac{2}{\pi} F_3(z) \qquad (0 < z < a) \tag{3.11}$$

whose solution is of the form

$$\varphi(t) = \frac{2}{\pi \sqrt{t}} \left[F_3(0) + t \int_0^{1/2\pi} F_3'(t \sin \theta) \, d\theta \right] \tag{3.12}$$

Substituting (3.10) into (3.12) and applying the equation

$$J_0(\lambda t) = \frac{2}{\pi} \int_0^{1/2\pi} \cos(\lambda t \sin \theta) \, d\theta$$

we obtain

$$\varphi(t) = t^{1/2} \int_0^{+\infty} \lambda g(\lambda) f_1(\lambda) J_0(\lambda t) \, d\lambda + \chi(t) \tag{3.13}$$

where

$$\chi(t) = \frac{2}{\pi \sqrt{t}} \left[F(0) + t \int_0^{1/2\pi} F'(t \sin \theta) \, d\theta \right] \tag{3.14}$$

Further, substituting (3.7) into (3.13) we obtain an integral equation that may be used to find $\varphi(t)$,

$$\varphi(t) = \int_0^a K(t, \tau) \varphi(\tau) \, d\tau + \chi(t) \tag{3.15}$$

where

$$K(t, \tau) = (t\tau)^{1/2} \int_0^{+\infty} \lambda g(\lambda) J_0(\lambda t) J_0(\lambda \tau) \, d\lambda \tag{3.16}$$

We combine the resulting solution (3.7) of the nonhomogeneous equations (3.5) with the solution of the homogeneous equations. We attempt to find a solution of equations (3.5) for $F(z) \equiv 0$ of the form

$$f_1(\lambda) = C_0[J_0(\lambda a) + f_3(\lambda)] \tag{3.17}$$

where C_0 is an arbitrary constant and $f_3(\lambda)$ is an unknown function. Substituting (3.17) into (3.5) we see that the function $f_3(\lambda)$ must satisfy the nonhomogeneous equations (3.5) with the right-hand side

$$F_4(z) = \int_0^{+\infty} g(\lambda) J_0(\lambda a) \sin \lambda z \, d\lambda \tag{3.18}$$

Hence

$$f_3(\lambda) = \int_0^a t^{1/2} \varphi_0(t) J_0(\lambda t) \, dt \tag{3.19}$$

where $\varphi_0(t)$ is found from the integral equation (3.15), (3.16) with absolute term (3.14), where the function $F(z)$ must be replaced by function (3.18). Thus, by virtue of (3.4), (3.7), (3.3), (3.17), and (3.19), the solution of the initial integral equations (3.1) is given by the formula

$$f(\lambda) = C_0 J_0(\lambda a) + \int_0^a t^{1/2} [\varphi(t) + C_0 \varphi_0(t)] J_0(\lambda t) \, dt + \frac{2}{\pi} \int_a^{+\infty} F_2(z) \cos \lambda z \, dz \tag{3.20}$$

where the functions $\varphi(t)$ and $\varphi_0(t)$ are found from the integral equation (3.15), (3.16). The absolute term $X(t)$ of equation (3.15) for finding the function $\varphi(t)$ is given by formulas (3.14) and (3.16); the absolute term $X(t)$ for finding $\varphi_0(t)$ is given by formula (3.14), where the function $F(z)$ must be replaced by $F_4(z)$ as obtained from formula (3.18).

It can be shown that if the function $g(\lambda)$ possesses the property A, the kernel (3.16) is a function summed with a square in the rectangle $0 \leqslant t \leqslant a$, $0 \leqslant \tau \leqslant a$. In addition, it is a function continuous in the large [3]. If $\alpha = 0$, then the kernel (3.16) is continuous on the aforementioned rectangle. Hence, if the functions $F_1(z)$ and $F_2(z)$ are such that the absolute term $X(t)$ is a function summed with a square on $[0, a]$, then equation (3.15) is a Fredholm integral equation of the second kind with a symmetrical kernel. If $X(t)$ is continuous in the interval $[0, a]$, the solution of integral equation (3.15) is continuous in that interval [3]. If $g(\lambda)$ possesses the property A and the solution of integral equation (3.15) is a function of restricted variation, it can be shown* that all operations involving alteration of the order of integration, taking of limits, and differentiation under the integral sign carried out above are valid, and that the solution of the Schloemilch integral equations (3.11) has a continuous derivative [2] on $[0, a]$.

We might point out that a formally different representation of the solution of nonhomogeneous dual integral equations (3.1) is given in [4, 5] for the case $F_2(z) = 0$.

The dual integral equations

* The proof is omitted in view of its cumbersome character.

$$\int_0^{+\infty} [1 - g(\lambda)]\, \lambda f(\lambda) \cos \lambda z\, d\lambda = \Phi(z) \qquad (0 < z < a)$$

$$\int_0^{+\infty} f(\lambda) \cos \lambda z\, d\lambda = F_2(z) \qquad (a < z < +\infty)$$

may be reduced to integral equations (3.1) considered above by formally integrating the first equation from 0 to z and introducing the notation

$$\int_0^z \Phi(t)\, dt = F_1(z)$$

We next consider some dual integral equations of the second type,

$$\begin{aligned} &\int_0^{+\infty} [1 - g(\lambda)]\, f(\lambda) \sin \lambda z\, d\lambda = F_1(z) \qquad (0 < z < a) \\ &\int_0^{+\infty} \lambda f(\lambda) \sin \lambda z\, d\lambda = F_2(z) \qquad (a < z < +\infty) \end{aligned} \tag{3.21}$$

By similar reasoning we deduce that the solution of these equations is given by the formula

$$f(\lambda) = \int_0^a t^{1/2} \varphi(t)\, J_0(\lambda t)\, dt + \frac{2}{\pi\lambda} \int_a^{+\infty} F_2(z) \sin \lambda z\, dz \tag{3.22}$$

where the function $\varphi(t)$ is found from an integral equation with a symmetrical kernel,

$$\varphi(t) = \int_0^a K(t, \tau)\, \varphi(\tau)\, d\tau + \chi(t) \qquad (0 \leqslant t \leqslant a) \tag{3.23}$$

$$K(t, \tau) = (t\tau)^{1/2} \int_0^{+\infty} \lambda g(\lambda)\, J_0(\lambda\tau)\, J_0(\lambda t)\, d\lambda \tag{3.24}$$

$$\chi(t) = \frac{2}{\pi \sqrt{t}}\, Q(t) + \left(\frac{2}{\pi}\right)^{1/2} t^{1/2} \int_0^{+\infty} \lambda^{1/2} g(\lambda)\, M(\lambda)\, J_0(\lambda t)\, d\lambda \tag{3.25}$$

$$M(\lambda) = \left(\frac{2}{\pi}\right)^{1/2} \lambda^{-1/2} \int_a^{+\infty} F_2(z) \sin \lambda z\, dz \tag{3.26}$$

$$\begin{aligned} Q(t) = \lim_{z \to 0} \left[F_1(z) - \left(\frac{2}{\pi}\right)^{1/2} \int_0^{+\infty} \lambda^{-1/2} M(\lambda) \sin \lambda z\, d\lambda\right] + \\ + t \int_0^t (t^2 - z^2)^{-1/2} \frac{d}{dz} \left[F_1(z) - \left(\frac{2}{\pi}\right)^{1/2} \int_0^{+\infty} M(\lambda) \frac{\sin \lambda z}{\sqrt{\lambda}}\, d\lambda\right] dz \end{aligned} \tag{3.27}$$

If $g(\lambda)$ possesses the property A and the functions $F_1(z)$ and $F_2(z)$ are such that the absolute term (3.25) is a function summed with a square in $[0, a]$, then equation (3.23) is a Fredholm equation.

A special case of dual equations (3.21) is considered in [6].

4. The first problem of a cylinder in torsion. Symmetrical case. Let us consider a mixed boundary value problem on the symmetrical torsion of an elastic cylinder $0 \leqslant r \leqslant R,\ -\infty < z < +\infty$ (Fig. 1).

We are given the displacement $v(R, z)$ in the interval $-a < x < a$ of the side surface $r = R$ and the tangential stresses $\tau_{r\varphi}(R, z)$ on the remaining portion of the side surface of the cylinder. The general torsion problem we are about to consider is a combination of two special cases: 1) symmetrical loading of the side surface relative to the plane $z = 0$ (Fig. 1); 2) antisymmetrical loading relative to the plane $z = 0$.

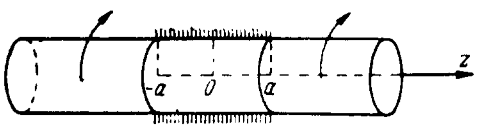

FIG. 1

We first solve the problem for the symmetrical case, mathematically formulated as follows: we are required to find the function $v(r, z)$ which satisfies the differential equation (1.1) inside the cylinder $0 \leqslant r \leqslant R,\ -\infty < z < +\infty$ and the conditions

$$v = \psi(z) \quad \text{for} \quad r = R, \quad -a < z < a \tag{4.1}$$

$$\tau_{r\varphi} = q(z) \quad \text{for} \quad r = R, \quad a < |z| < +\infty \tag{4.2}$$

on its surface.

$\psi(z)$ and $q(z)$ in these expressions are even functions; the projection of the moment on the z-axis must equal zero

$$4\pi R^2 \int_0^{+\infty} \tau_{r\varphi}(R, z)\, dz = 0 \tag{4.3}$$

If the cylinder is fixed in the interval $-a < z < a$ of the side surface, then $\psi(z) = 0$.

The function $q(z)$ is considered representable as a Fourier integral,

$$q(z) = \int_0^{+\infty} q_1(\lambda) \cos \lambda z\, d\lambda, \qquad q_1(\lambda) = \frac{2}{\pi} \int_a^{+\infty} q(z) \cos \lambda z\, dz \tag{4.4}$$

We attempt to solve the problem posed in the form of formulas (1.5), setting

$$A_1(\lambda) = 0 \tag{4.5}$$

Instead of boundary condition (4.1) we consider the boundary condition for the partial derivative

$$\partial v / \partial z = \psi (z) \quad \text{for } r = R, \quad -a < z < a \tag{4.6}$$

Satisfying boundary conditions (4.6) and (4.2), we obtain the following dual integral equations in the function $A_2(\lambda)$:

$$\begin{aligned} &\int_0^{+\infty} \lambda^2 A_2(\lambda) I_1(\lambda R) \sin \lambda z \, d\lambda = \psi'(z) \qquad (0 < z < a) \\ &\int_0^{+\infty} \lambda^2 A_2(\lambda) I_2(\lambda R) \cos \lambda z \, d\lambda = \frac{1}{G} q(z) \qquad (a < z < +\infty) \end{aligned} \tag{4.7}$$

If we set

$$\begin{aligned} A_2(\lambda) &= \frac{f(\lambda)}{\lambda^2 I_2(\lambda R)}, \qquad F_1(z) = \psi'(z), \qquad F_2(z) = \frac{1}{G} q(z) \\ 1 - g(\lambda) &= \frac{I_1(\lambda R)}{I_2(\lambda R)}, \qquad g(\lambda) = 1 - \frac{I_1(\lambda R)}{I_2(\lambda R)} \end{aligned} \tag{4.8}$$

equations (4.7) reduce to the form (3.1). Hence, by virtue of (3.20), their solution is given by the formula

$$A_2(\lambda) = \frac{1}{\lambda^2 I_2(\lambda R)} \left\{ C_0 J_0(\lambda a) + \int_0^a t^{1/2} [\varphi(t) + C_0 \varphi_0(t)] J_0(\lambda t) \, dt + f_2(\lambda) \right\} \tag{4.9}$$

where $\varphi(t)$ and $\varphi_0(t)$ are found from integral equation (3.15), (3.16) and (3.14). The function $F(z)$ appearing in the expression for the absolute term (3.14) for finding $\varphi(t)$ has the form

$$F(z) = \psi'(z) - \int_0^{+\infty} \frac{I_1(\lambda R)}{I_2(\lambda R)} f_2(\lambda) \sin \lambda z \, d\lambda$$

for finding $\varphi_0(t)$ it may be written as

$$F(z) = \int_0^{+\infty} g(\lambda) J_0(\lambda a) \sin \lambda z \, d\lambda$$

where $g(\lambda)$ is given by formula (4.8), and

$$f_2(\lambda) = \frac{2}{\pi G} \int_a^{+\infty} q(z) \cos \lambda z \, dz \qquad (0 \leqslant \lambda < +\infty) \tag{4.10}$$

It is easy to show that

$$\lim_{\lambda \to 0} \lambda g(\lambda) = -\frac{4}{R}, \qquad \lim_{\lambda \to +\infty} \lambda^p \left[\lambda g(\lambda) + \frac{3}{2R} \right] = 0 \qquad (0 < p < 1)$$

i.e., the function $g(\lambda)$ possesses the property A. Thus, the kernel (3.16) in this problem is a Fredholm kernel.

Let us find the distribution of the stresses $\tau_{r\varphi}$ on the fixed segement $-a < z < a$ of the side surface. By virtue of (4.5), expressions (1.5) yield

$$\tau_{r\varphi}(R, z) = G \int_0^{+\infty} \lambda^2 A_2(\lambda) I_2(\lambda R) \cos \lambda z \, d\lambda \tag{4.11}$$

Substituting (4.9) into (4.11) and making use of integral (3.8) and the inversion formula for (4.10), we obtain

$$\tau_{r\varphi}(R, z) = G \left\{ \frac{C_0}{\sqrt{a^2 - z^2}} + \int_z^a \frac{t^{1/2} [\varphi(t) + C_0 \varphi_0(t)]}{\sqrt{t^2 - z^2}} dt \right\} \qquad (-a < z < a) \tag{4.12}$$

Hence, the shear stresses on the fixed portion of the side surface may be expressed directly in terms of the functions $\varphi(t)$ and $\varphi_0(t)$. Clearly, the shear stresses $\tau_{r\varphi}$ become infinite at the points $z = \pm a$ of the side surface. Their distribution law is close to the distribution law for normal stresses under a die [7].

In order to complete the solution of the problem, we must determine the constant C_0. We can find it using equilibrium condition (4.3), which together with boundary condition (4.6) is equivalent to boundary condition (4.1). Substituting (4.12) and (4.2) into (4.3) and computing the integrals, we obtain

$$C_0 \left[1 + \int_0^a t^{1/2} \varphi_0(t) \, dt \right] = - \int_0^a t^{1/2} \varphi(t) \, dt - \frac{M_k}{2\pi^2 R^2 G} \qquad \left(M_k = 4\pi R^2 \int_a^{+\infty} q(z) \, dz \right)$$

where M_k is the torsional moment comprised of external stresses (4.2) applied to the side surface of the cylinder.

5. The first problem of a cylinder in torsion. Antisymmetrical problem. Let us write out the boundary conditions of the problem, assuming that the segment $-a < z < a$ of the side surface is fixed (Fig. 2):

$$v = 0 \quad \text{for} \quad r = R, \ -a < z < a \tag{5.1}$$

$$\tau_{r\varphi} = F_2(z) \quad \text{for} \quad r = R, \ a < |z| < +\infty \tag{5.2}$$

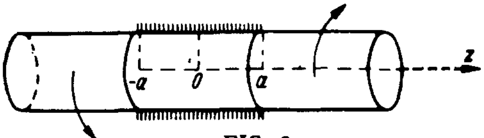

FIG. 2

where $F_2(z)$ is an odd function. The shear stresses applied to the segment of the side surface extending from a to $+\infty$ produce the torsional moment

$$M_k = 2\pi R^2 \int_a^{+\infty} F_2(z) \, dz$$

The function $F_2(z)$ is assumed to be representable as a Fourier integral,

$$F_2(z) = \int_0^{+\infty} f_2(\lambda) \sin \lambda z \, d\lambda, \qquad f_2(\lambda) = \frac{2}{\pi} \int_a^{+\infty} F_2(z) \sin \lambda z \, dz \tag{5.3}$$

Clearly,

$$\int_0^{+\infty} f_2(\lambda) \sin \lambda z \, d\lambda = 0 \qquad (0 < z < a) \tag{5.4}$$

We attempt to solve this problem in the form of formulas (1.5), where we set

$$A_2(\lambda) = 0 \tag{5.5}$$

Satisfying boundary conditions (5.1) and (5.2), we obtain dual integral equations in the unknown function $A_1(\lambda)$,

$$\begin{gathered} \int_0^{+\infty} \lambda A_1(\lambda) I_1(\lambda R) \sin \lambda z \, d\lambda = 0 \qquad (0 < z < a) \\ G \int_0^{+\infty} \lambda^2 A_1(\lambda) I_2(\lambda R) \sin \lambda z \, d\lambda = F_2(z) \qquad (a < z < +\infty) \end{gathered} \tag{5.6}$$

If we set

$$A_1(\lambda) = \frac{f(\lambda)}{G\lambda I_2(\lambda R)}, \qquad \frac{I_1(\lambda R)}{I_2(\lambda R)} = 1 - g(\lambda) \tag{5.7}$$

equations (5.6) are reduced to the form (3.21) for $F_1(z) = 0$. By virtue of (3.22), their solution is therefore given by the formula

$$A_1(\lambda) = \frac{1}{G\lambda I_2(\lambda R)} \left[\int_0^a t^{1/2} \varphi(t) J_0(\lambda t) \, dt + \frac{2}{\pi \lambda} \int_a^{+\infty} F_2(z) \sin \lambda z \, dz \right] \tag{5.8}$$

where $\varphi(t)$ is found from the integral equation (3.23), (3.24), and (3.25). Since the function $g(\lambda)$ given by formula (5.7) coincides with function (4.8), kernel (3.24) is a Fredholm kernel.

Let us find the distribution of shear stresses $\tau_{r\varphi}$ on the fixed segment $0 \leqslant z < a$ of the side surface. To this end, we transform expression (5.8) using (5.3) and the formula

$$J_0(\lambda t) = \frac{1}{\lambda} \frac{dJ_1(\lambda t)}{dt} + \frac{1}{\lambda t} J_1(\lambda t)$$

We obtain

$$A_1(\lambda) = \frac{1}{G\lambda^2 I_2(\lambda R)} \left\{ a^{1/2} \varphi(a) J_1(\lambda a) - \right. \tag{5.9}$$

$$-\int_0^a\left[t^{1/2}\varphi'(t)-\frac{1}{2}t^{-1/2}\varphi(t)\right]J_1(\lambda t)\,dt+f_2(\lambda)\Big\} \tag{5.9}$$

Substituting (5.9) into expression (1.5) for $\tau_{r\varphi}$ and taking into account (5.4) and (5.5), we obtain

$$\tau_{r\varphi}(R,\ z)=\frac{a^{-1/2}\varphi(a)\,z}{\sqrt{a^2-z^2}}-z\int_z^a\left[t^{1/2}\varphi'(t)-\frac{1}{2}t^{-1/2}\varphi(t)\right]\frac{dt}{t\sqrt{t^2-z^2}} \qquad (0<z<a)$$

where we have made use of the integral

$$\int_0^{+\infty}J_1(\lambda t)\sin\lambda z\,d\lambda=\begin{cases} z\cdot t^{-1}(t^2-z^2)^{-1/2}, & \text{if}\quad 0<z<t\\ 0, & \text{if}\quad t<z<+\infty\end{cases}$$

Clearly, the shear stresses become infinite at the point $z=\pm a$ of the side surface.

6. The second problem of a cylinder in torsion. Symmetrical case. Let the shear stresses $\tau_{r\varphi}(R,\ z)$ be given on the segment $-a<z<a$ of the side surface of the cylinder (Fig. 3); the displacement $v\,(R,\ z)$ is given on the remaining portion of the side surface. In the general case this problem is a combination of the symmetrical and antisymmetrical cases of loading of the side surface relative to the plane $z=0$.

The boundary conditions for the symmetrical case may be written as

$$\tau_{r\varphi}=q\,(z)\quad \text{for}\quad r=R,\ -a<z<a \tag{6.1}$$

$$v=F_2\,(z)\quad \text{for}\quad r=R,\ a<|z|<+\infty \tag{6.2}$$

where $q\,(z)$ and $F_2\,(z)$ are even functions. The projection of the moment on the z-axis must equal zero,

$$4\pi R^2\int_0^{+\infty}\tau_{r\varphi}(R,\ z)\,dz=0$$

The function $F_2\,(z)$ is considered to be representable as a Fourier integral,

$$F_2(z)=\int_0^{+\infty}f_2(\lambda)\cos\lambda z\,d\lambda,\qquad f_2(\lambda)=\frac{2}{\pi}\int_a^{+\infty}F_2(z)\cos\lambda z\,dz \tag{6.3}$$

We attempt to solve the problem in the form of formulas (1.5), where we set $A_1(\lambda)=0$.

Satisfying boundary conditions (6.1) and (6.2), we obtain dual integral equation for finding the unknown function $A_2(\lambda)$,

$$G\int_0^{+\infty}\lambda^2A_2(\lambda)\,I_2(\lambda R)\cos\lambda z\,d\lambda=q\,(z)\qquad (0<z<a)$$

$$\int_0^{+\infty}\lambda A_2(\lambda)\,I_1(\lambda R)\cos\lambda z\,d\lambda=F_2(z)\qquad (a<z<+\infty) \tag{6.4}$$

By integrating the first of these equations from 0 to z and setting

$$A_2(\lambda)=\frac{f(\lambda)}{\lambda I_1(\lambda R)}, \qquad g(\lambda)=1-\frac{I_2(\lambda R)}{I_1(\lambda R)}, \qquad F_1(z)=\frac{1}{G}\int_0^z q(z)\,dz \tag{6.5}$$

we reduce equation (6.4) to the form (3.1). Their solution is given by formula (3.20).

Setting $C_0=0$ in formula (3.20), we obtain the following expression for the solution of equations (6.4):

$$A_2(\lambda)=\frac{1}{\lambda I_1(\lambda R)}\left\{\int_0^a t^{1/2}\varphi(t)J_0(\lambda t)\,dt+\frac{2}{\pi}\int_a^{+\infty}F_2(z)\cos\lambda z\,dz\right\} \tag{6.6}$$

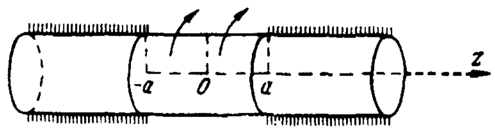

FIG. 3

where $\varphi(t)$ is found from integral equation (3.15), (3.16), and (3.14). The function $F(z)$ appearing in the absolute term (3.14) may be written as

$$F(z)=\frac{1}{G}\int_0^z q(z)\,dz-\int_0^{+\infty}\frac{I_2(\lambda R)}{I_1(\lambda R)}f_2(\lambda)\sin\lambda z\,d\lambda$$

where $f_2(\lambda)$ is given by formula (6.3).

It is easy to show that

$$\lim_{\lambda\to 0}\lambda g(\lambda)=\frac{5}{2R}, \qquad \lim_{\lambda\to+\infty}\lambda^p\left[\lambda g(\lambda)-\frac{3}{2R}\right]=0 \qquad (0<p<1)$$

i.e., the function $g(\lambda)$ possesses the property A. Kernel (3.16) in this problem is therefore a Fredholm kernel.

Let us find the distribution of shear stresses on the side surface in the case where the side surface outside the segment $-a<z<a$ is fixed. In this case $F_2(z)=0$, and by virtue of the identity

$$\lambda J_0(\lambda t)=\frac{dJ_1(\lambda t)}{dt}+\frac{1}{t}J_1(\lambda t)$$

formula (6.6) is easily reduced to the form

$$A_2(\lambda)=\frac{1}{\lambda^2 I_1(\lambda R)}\left[a^{1/2}\varphi(a)J_1(\lambda a)+\varphi_1(\lambda)\right] \tag{6.7}$$

where

$$\varphi_1(\lambda)=\int_0^a\{t^{-1/2}\varphi(t)-[t^{1/2}\varphi(t)]'\}J_1(\lambda t)\,dt$$

Substituting (6.7) into expression (1.5) for $\tau_{r\varphi}$, we obtain

$$\tau_{r\varphi}(R, z) = \frac{Ga^{3/2}\varphi(a)}{\sqrt{z^2-a^2}\,[z+\sqrt{z^2-a^2}]} - G\int_0^{+\infty} g(\lambda)\lambda f(\lambda)\cos\lambda z\, d\lambda +$$

$$+ G\int_0^a \frac{t\{t^{-1/2}\varphi(t) - [t^{1/2}\varphi(t)]'\}}{\sqrt{z^2-t^2}\,[z+\sqrt{z^2-t^2}]}\,dt \qquad (a < z < +\infty)$$

where we have used the integral

$$\int_0^{+\infty} J_1(\lambda t)\cos\lambda z\, d\lambda = \frac{t}{\sqrt{z^2-t^2}\,[z+\sqrt{z^2-t^2}]} \qquad (0 < t < z)$$

The functions $g(\lambda)$ and $f(\lambda)$ are given by formulas (6.5) and (6.6).

7. The second problem of a cylinder in torsion. Antisymmetrical case. The boundary conditions of this problem may be written as (Fig. 4)

$$\tau_{r\varphi} = q(z) \quad \text{for} \quad r = R, \; -a < z < a \tag{7.1}$$

$$v = h(z) \quad \text{for} \quad r = R, \; a < |z| < +\infty \tag{7.2}$$

where $q(z)$ and $h(z)$ are odd functions. We assume that the derivative $h'(z) = F_2(z)$ is representable as a Fourier integral,

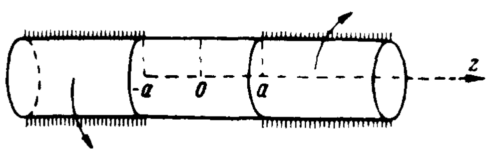

FIG. 4

$$h'(z) = F_2(z) = \int_0^{+\infty} f_2(\lambda)\cos\lambda z\, d\lambda,$$

$$f_2(\lambda) = \frac{2}{\pi}\int_a^{+\infty} h'(z)\cos\lambda z\, dz \tag{7.3}$$

We attempt to solve this problem in the form of formulas (1.5), where we set

$$A_2(\lambda) = 0$$

Instead of boundary condition (7.2) we consider the boundary condition for the partial derivative

$$\frac{\partial v}{\partial z} = h'(z) = F_2(z) \quad \text{for} \quad r = R, \quad a < |z| < +\infty \tag{7.4}$$

and the additional condition

$$v = (R, z_0) = h(z_0) \qquad (z_0 > a) \tag{7.5}$$

which consists in the equality of the displacement $v(r, z)$ being sought to its given value at some point $z_0 > a$ of the side surface $r = R$. Conditions (7.4) and (7.5) are clearly equivalent to boundary condition (7.2).

Satisfying boundary conditions (7.1) and (7.4), we obtain dual integral equations for finding the unknown function $A_1(\lambda)$,

$$G\int_0^{+\infty} \lambda^2 A_1(\lambda) I_2(\lambda R)\sin\lambda z d\lambda = q(z) \qquad (0 < z < a)$$

$$\int_0^{+\infty} \lambda^2 A_1(\lambda) I_1(\lambda R)\cos\lambda z d\lambda = F_2(z) \qquad (a < z < +\infty)$$

These equations may be reduced to the form of equations (3.1) by setting

$$A_1(\lambda) = \frac{f(\lambda)}{\lambda^2 I_1(\lambda R)}, \qquad \frac{I_2(\lambda R)}{I_1(\lambda R)} = 1 - g(\lambda), \qquad \frac{1}{G}q(z) = F_1(z) \tag{7.6}$$

For this reason their solution (by virtue of (3.20), (7.4), and (7.3)) is given by the formula

$$A_1(\lambda) = \frac{1}{\lambda^2 I_1(\lambda R)}\left\{C_0 J_0(\lambda a) + \int_0^a t^{1/2}[\varphi(t) + C_0\varphi_0(t)] J_0(\lambda t)\,dt + f_2(\lambda)\right\} \tag{7.7}$$

where C_0 is an arbitrary constant. The functions $\varphi(t)$ and $\varphi_0(t)$ are found from integral equation (3.15), (3.16), and (3.14). The function $F(z)$ appearing in the expression for the absolute term (3.14) for finding $\varphi(t)$ may be written as

$$F(z) = \frac{1}{G}q(z) - \int_0^{+\infty}\frac{I_2(\lambda R)}{I_1(\lambda R)} f_2(\lambda)\sin\lambda z d\lambda$$

for finding $\varphi_0(t)$ we have the expression

$$F(z) = \int_0^{+\infty} g(\lambda) J_0(\lambda a)\sin\lambda z d\lambda, \qquad F(0) = 0$$

where $g(\lambda)$ and $f_2(\lambda)$ are given by formulas (7.6) and (7.3).

The functions $g(\lambda)$ as expressed by formulas (7.6) and (6.5) coincide. This means that kernel (3.16) of integral equation (3.15) in this problem is a Fredholm kernel.

The constant C_0 may be determined from condition (7.5). Substituting the value of $A_1(\lambda)$ given by formula (7.7) into expression (1.5) for $v(r, z)$, and then $v(r, z)$ into equation (7.5), we obtain the following expression for C_0:

$$C_0\left[1 + \int_0^a t^{1/2}\varphi_0(t)\,dt\right] = \tag{7.8}$$

$$= \frac{2}{\pi} h(z_0) - \int_0^a t^{1/2}\varphi(t)\,dt - \frac{2}{\pi}\int_0^{+\infty} \lambda^{-1} f_2(\lambda)\sin\lambda z_0 d\lambda \qquad (7.8)$$

where we have used the integral

$$\int_0^{+\infty} \lambda^{-1} J_0(\lambda t)\sin\lambda z\,d\lambda = \frac{\pi}{2} \qquad (0 < t < z)$$

Any number larger than a can be taken as z_0 in formula (7.8).

Let us quote the formula for the distribution of shear stresses on the segment $a < z < +\infty$ of the side surface of the cylinder for the special case of the boundary conditions of the problem being considered. Namely, let

$$h(z) = \begin{cases} h_0, & \text{if } a < z < +\infty \\ -h_0, & \text{if } -\infty < z < -a \end{cases}$$

It is then easy to show that

$$\tau_{r\varphi}(R, z) = \frac{C_0 G}{\sqrt{z^2 - a^2}} + G\int_0^a \frac{t^{1/2}[\varphi(t) + C_0\varphi_0(t)]}{\sqrt{z^2 - t^2}}\,dt -$$

$$- G\int_0^{+\infty} g(\lambda) f(\lambda)\sin\lambda z d\lambda \qquad (a < z < +\infty)$$

where $f(\lambda)$ and C_0 are given by formulas (7.6), (7.7), and (7.8), where

$$f_2(\lambda) = 0$$

We note in conclusion that the same method may be used to solve analogous problems involving the torsion of a hollow cylinder.

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